ON NONEMPTINESS OF NEWTON STRATA IN THE B_{dR}^+-GRASSMANNIAN FOR GL_n

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ABSTRACT. We study the Newton stratification in the B_{dR}^+-Grassmannian for GL_n associated to an arbitrary (possibly nonbasic) element of $B(\mathrm{GL}_n)$. Our main result classifies all nonempty Newton strata in an arbitrary minuscule Schubert cell. For a large class of elements in $B(\mathrm{GL}_n)$, our classification is given by some explicit conditions in terms of Newton polygons. For the proof, we proceed by induction on n using a previous result of the author that classifies all extensions of two given vector bundles on the Fargues-Fontaine curve.

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1. INTRODUCTION

1.1. Motivation and main result.

The B_{dR}^+-Grassmannian is an analogue of the affine Grassmannian in p-adic geometry. It was introduced by Caraiani-Scholze [CS17] to study the cohomology of certain Shimura varieties, and also used by Scholze-Weinstein [SW20] as a crucial tool for the construction of local Shimura varieties. In addition, it played a fundamental role in the work of Fargues-Scholze [FS21] on the geometrization of the local Langlands correspondence via the geometric Satake equivalence for p-adic groups.

The main objective of this paper is to study a natural stratification of the B_{dR}^+-Grassmannian known as the *Newton stratification*, which we briefly describe now. Let us fix a connected reductive group G over a finite extension E of $\mathbb{Q}_p$. We write Gr_G for the B_{dR}^+-Grassmannian for G, and $\mathrm{Gr}_{G,\mu}$ for the Schubert cell associated to a dominant cocharacter μ of G. For an complete algebraically closed extension C of E, we have

$$\mathrm{Gr}_G(C) = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+) \quad \text{and} \quad \mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+)$$

where B_{dR} is the p-adic de Rham period ring with valuation ring B_{dR}^+, residue field C and a fixed uniformizer t. The Cartan decomposition for G induces a decomposition

$$\mathrm{Gr}_G = \bigsqcup_{\mu \in X_*(T)^+} \mathrm{Gr}_{G,\mu}$$

where $X_*(T)^+$ denotes the set of all dominant cocharacters of G. Moreover, each Schubert cell $\mathrm{Gr}_{G,\mu}$ is related to the (diamond of the) p-adic flag variety $\mathscr{F}\ell(G,\mu)$ via a natural Bialynicki-Birula map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu} \longrightarrow \mathscr{F}\ell(G,\mu),$$

which is an isomorphism if μ is minuscule. In order to define the Newton stratification on Gr_G and its Schubert cells, we consider the stack Bun_G of G-bundles on the Fargues-Fontaine curve X. By the result of Fargues [Far20], the topological space $|\mathrm{Bun}_G|$ of Bun_G is in natural bijection with the set $B(G)$ of Frobenius-conjugacy classes of elements of $G(\breve{E})$, where $\breve{E}$ as usual denotes the p-adic completion of the maximal unramified extension of E. Given an element $b \in B(G)$, we write $\mathcal{E}_b$ for the corresponding G-bundle on X. The theorem of Beauville-Laszlo [BL95] implies that a G-bundle on the Fargues-Fontaine curve is specified by the gluing data of the trivial G-bundles on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$ and $X - \infty$, where ∞ is a fixed closed point on X with residue field C and completed local ring B_{dR}^+. If we fix $b \in B(G)$, for every point $x \in \mathrm{Gr}_G(C)$ we can modify the gluing data for $\mathcal{E}_b$ by x to obtain a new G-bundle $\mathcal{E}_{b,x}$. We thus obtain a map

$$\mathrm{Newt}_b : \mathrm{Gr}_G(C) \longrightarrow B(G)$$

which maps each $x \in \mathrm{Gr}_G(C)$ to the element $b' \in B(G)$ corresponding to $\mathcal{E}_{b,x}$. For each Schubert cell $\mathrm{Gr}_{G,\mu}$, the Newton stratification associated to $b \in B(G)$ is a decomposition into subdiamonds

$$\mathrm{Gr}_{G,\mu} = \bigsqcup_{b' \in B(G)} \mathrm{Gr}_{G,\mu,b}^{b'}$$

where $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ is the preimage of b' in $\mathrm{Gr}_{G,\mu}(C)$ under the map Newt_b.

The Newton stratification of minuscule Schubert cells was originally introduced in the aforementioned work of Caraiani-Scholze [CS17] as a key tool for studying the fibers of the Hodge-Tate period map. It has also been used as a pivotal tool for studying the p-adic period domain by many authors, such as Chen-Fargues-Shen [CFS21], Shen [She23], Chen [Che20], Viehmann [Vie24], Nguyen-Viehmann [NV23], and Chen-Tong [CT22].

For the trivial element $b = 1$, a result of Rapoport [Rap18] shows that the Newton stratum $\mathrm{Gr}_{G,\mu,b}^{b'}$ is nonempty if and only if b' is an element of the set $B(G, -\mu)$ defined by Kottwitz [Kot85]. When b is basic, meaning that $\mathcal{E}_b$ is semistable, Chen-Fargues-Shen [CFS21] and Viehmann [Vie24] extends the result of Rapoport to parametrize all nonempty Newton strata by a generalized Kottwitz set. However, for a general element $b \in B(G)$, no explicit parametrization is known for nonempty Newton strata in an arbitrary Schubert cell.

In order to explain our main result, which classifies all nonempty Newton strata in the Schubert cell $\mathrm{Gr}_{G,\mu}$ for $G = \mathrm{GL}_n$ and a minuscule cocharacter μ, we need to set up some notations. Let us recall that, as observed by Kottwitz [Kot85], the set $B(\mathrm{GL}_n)$ is naturally identified with the set of concave polygons on the interval $[0, n]$ with rational slopes and integer breakpoints, where a polygon refers to a continuous piecewise linear function whose graph passes through the origin. Given an element $b \in B(\mathrm{GL}_n)$, we write $\nu(b)$ for the corresponding polygon and often regard it as a tuple of rational numbers $(\nu_1(b), \cdots, \nu_n(b))$ where $\nu_i(b)$ denotes the slope of $\nu(b)$ on the interval $[i-1, i]$. We may also represent the dominant cocharacter μ of GL_n as an n-tuple of descending integers $(\mu_1, \cdots, \mu_n)$ and regard it as a concave polygon on $[0, n]$ whose slope on $[i-1, i]$ is μ_i.

Given two arbitrary elements $b,\ b' \in B(\mathrm{GL}_n)$, our main result gives an inductive criterion for the nonemptiness of the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$. Let us provide a brief description of the inductive criterion here and refer the readers to Theorem 3.1.12 for a precise statement. If b is basic, meaning that $\nu(b)$ is a line segment, the desired classification is given by the aforementioned results of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. If b is not basic, we have unique elements $a \in B(\mathrm{GL}_m)$ and $c \in B(\mathrm{GL}_{n-m})$ for some integer m such that $\nu(a)$ and $\nu(c)$ together form a partition of $\nu(b)$ with $\nu(a)$ being the line segment in $\nu(b)$ of maximum slope. The key observation for our main result is that $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty if and only if there exist $a' \in B(\mathrm{GL}_m)$ and $c' \in B(\mathrm{GL}_{n-m})$ with the following properties:

(i) The Newton strata $\mathrm{Gr}^{a'}_{\mathrm{GL}_m,\mu_1,a}$ and $\mathrm{Gr}^{c'}_{\mathrm{GL}_{n-m},\mu_2,c}$ are not empty for some minuscule cocharacters μ_1 of GL_m and μ_2 of GL_{n-m}.

(ii) The vector bundle $\mathcal{E}_{b'}$ arises as an extension of $\mathcal{E}_{c'}$ by $\mathcal{E}_{a'}$; in other words, there exists a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{E}_{a'} \longrightarrow \mathcal{E}_{b'} \longrightarrow \mathcal{E}_{c'} \longrightarrow 0.$$

The cocharacters μ_1 and μ_2 in the property (i) are uniquely determined by a' and c'. Moreover, the property (i) imposes explicit bounds on the slopes in $\nu(a')$ and $\nu(c')$, and consequently yields a finite list of candidates for (a', c'). For each candidate, we can check the property (ii) by a previous result of the author [Hon22]. Then for each candidate with the property (ii), we can inductively proceed to check the property (i); indeed, if $\nu(b)$ has r distinct slopes, then $\nu(c)$ has $r-1$ distinct slopes while $\nu(a)$ is a line segment by construction.

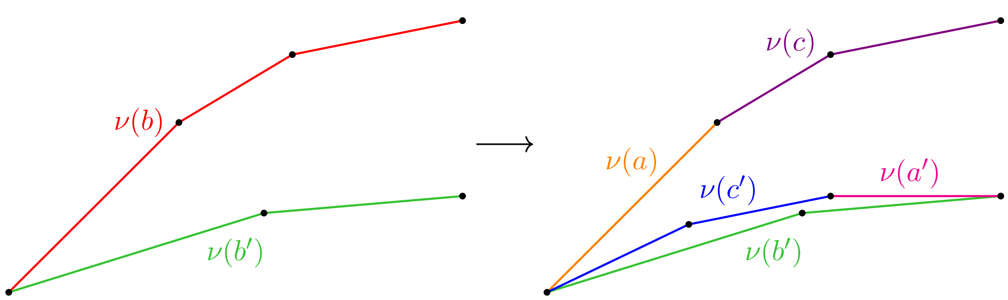

FIGURE 1. Illustration of the inductive criterion

For a concrete example, we illustrate how our inductive criterion shows the nonemptiness of the stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_8,\mu,b}$ with

$$\nu(b) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right),$$
$$\nu(b') = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, 0\right),$$
$$\mu = (1, 1, 1, 1, 0, 0, 0, 0).$$

The elements $a \in B(\mathrm{GL}_3)$ and $c \in B(\mathrm{GL}_5)$ are given by

$$\nu(a) = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right) \quad \text{and} \quad \nu(c) = \left(\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}\right).$$

We apply the inductive criterion with $a' \in B(\mathrm{GL}_3)$ and $c' \in B(\mathrm{GL}_5)$ given by

$$\nu(a') = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) \quad \text{and} \quad \nu(c') = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right).$$

Indeed, the nonemptiness of the stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_8,\mu,b}$ follows from the following statements:

- $\mathcal{E}_{b'}$ arises as an extension of $\mathcal{E}_{c'}$ by $\mathcal{E}_{a'}$.
- $\mathrm{Gr}^{a'}_{\mathrm{GL}_3,\mu_1,a}$ with $\mu_1 = (1,1,1)$ and $\mathrm{Gr}^{c'}_{\mathrm{GL}_5,\mu_2,c}$ with $\mu_2 = (1,0,0,0,0)$ are not empty.

For the second statement, we note that a and c are basic for $\nu(a)$ and $\nu(c)$ being line segments.

A special case of our main result reduces to a noninductive criterion as follows:

Theorem 1.1.1. *Let μ be a minuscule dominant cocharacter of $G = \mathrm{GL}_n$ represented by an n-tuple with entries 0 and 1. Given two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ such that the difference between any two distinct slopes in $\nu(b)$ is greater than 1, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if the following conditions are satisfied:*

(i) The polygon $\nu(b')$ lies below the polygon $\nu(b) + \mu^*$ with the same endpoints, where μ^* denotes the unique dominant cocharacter of GL_n in the conjugacy class of μ^{-1}.

(ii) We have inequalities

$$\nu_i(b') \leq \nu_i(b) \leq \nu_i(b') + 1 \quad \text{for } i = 1, \cdots, n.$$

(iii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.

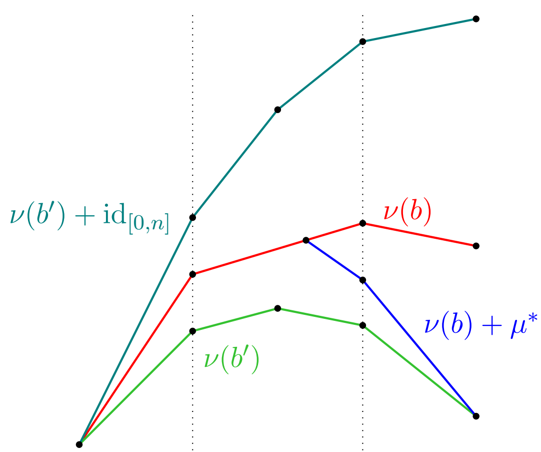

FIGURE 2. Illustration of the conditions in Theorem 1.1.1

The condition (i) is in fact equivalent to having b' in the generalized Kottwitz set considered by Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. When b is basic, the condition (i) also implies the conditions (ii) and (iii). Hence when b is basic Theorem 1.1.1 agrees with the aforementioned result of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24].

The hypothesis on the cocharacter μ having entries 0 and 1 is insignificant; indeed, without this assumption we still get a similar statement by a simple reduction technique as stated in Proposition 3.1.6. On the other hand, the hypothesis on the slopes in $\nu(b)$ is crucial. For the general case, the conditions (i) and (ii) are still necessary but not sufficient.

1.2. Outline of the proof.

Given a vector bundle $\mathcal{E}$ on the Fargues-Fontaine curve X, its *minuscule effective modification at* ∞ of degree d refers to an injective bundle map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $C^{\oplus d}$. The Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞. We thus wish to classify all minuscule effective modifications of $\mathcal{E}_b$ at ∞. If b is basic, the desired classification is given by the aforementioned results of Chen-Fargues-Shen [CFS21] and Viehmann [Vie24]. Let us now assume that b is not basic. We can find a direct sum decomposition

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad \text{with} \quad a \in B(\mathrm{GL}_m) \text{ and } c \in B(\mathrm{GL}_{n-m})$$

where a is basic such that $\nu(a)$ equals the line segment in $\nu(b)$ of maximum slope. For every minuscule effective modification $\iota : \mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞, the above decomposition extends to a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_a \arrow[r] & \mathcal{E}_b \arrow[r] & \mathcal{E}_c \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}_{a'} \arrow[r] \arrow[u, hook, "\alpha"] & \mathcal{E}_{b'} \arrow[r] \arrow[u, hook, "\beta"] & \mathcal{E}_{c'} \arrow[r] \arrow[u, hook, "\gamma"] & 0
\end{tikzcd}$$

where α and γ are also minuscule effective modifications at ∞. Conversely, given such a commutative diagram we apply a result of Chen-Tong [CT22] to observe that α and γ can be adjusted so that β is a minuscule effective modification at ∞. Then we use a previous result of the author [Hon22] to classify all vector bundles $\mathcal{E}_{a'}$ and $\mathcal{E}_{c'}$ that fit into such a commutative diagram, and consequently proceed by induction to obtain the desired classification.

1.3. Notations and conventions.

Throughout the paper, we fix the following data:

- E is a finite extension of $\mathbb{Q}_p$.
- C is a complete and algebraically closed extension of E.
- G is a reductive group over E with Borel subgroup B and maximal torus $T \subseteq B$.

We also retain the following notations:

- $\breve{E}$ is the p-adic completion of the maximal unramified extension of E.
- $B(G)$ is the set of Frobenius-conjugacy classes of elements of $G(\breve{E})$.
- $X_*(T)^+$ is the set of all dominant cocharacters of G.

In addition, we use the following standard notations:

- Given a valued field K, we write $\mathcal{O}_K$ for its valuation ring.
- Given a ringed space S, we write $\mathcal{O}_S$ for its structure sheaf.
- Given a perfectoid ring R, we write $R^\flat$ for its tilt and R° for its subring of power bounded elements.
- Given a perfect $\mathbb{F}_p$-algebra A, we write $W(A)$ for the ring of Witt vectors over A.

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2. Preliminaries

In this section, we review some basic facts about the B_{dR}^{+}-Grassmannian and G-bundles on the Fargues-Fontaine curve.

2.1. The B_{dR}^{+}-Grassmannian.

Proposition 2.1.1 ([Fon82, Proposition 2.4], [KL15, Lemma 3.6.3])**.** *Let R be a perfectoid algebra over C. There exists a natural surjective homomorphism $W(R^{\circ\flat}) \twoheadrightarrow R^{\circ}$ whose kernel is a principal ideal of $W(R^{\circ\flat})$.*

Definition 2.1.2. Let R be a perfectoid algebra over C. Choose a generator t of the kernel of the map $W(R^{\circ\flat}) \twoheadrightarrow R^{\circ}$ in Proposition 2.1.1. We write $B_{\mathrm{dR}}^{+}(R)$ for the t-adic completion of $W(R^{\circ\flat})[1/p]$, and define the *de Rham period ring* associated to R by $B_{\mathrm{dR}}(R) := B_{\mathrm{dR}}^{+}(R)[1/t]$.

Proposition 2.1.3 ([Fon82, Proposition 2.17])**.** *The ring $B_{\mathrm{dR}}(C)$ is a discretely valued field with valuation ring $B_{\mathrm{dR}}^{+}(C)$ and residue field C.*

We will henceforth write $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$ and $B_{\mathrm{dR}}^{+} := B_{\mathrm{dR}}^{+}(C)$. We also fix a uniformizer t of B_{dR} in light of Proposition 2.1.3.

Definition 2.1.4. The B_{dR}^{+}-*Grassmannian* is the functor Gr_G that associates to each perfectoid affinoid algebra (R, R^{+}) over C the set of pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}^{+}(R))$ and a trivialization β of $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}(R))$.

Proposition 2.1.5 ([SW20, Proposition 19.1.2])**.** *There exists a natural identification*

$$\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^{+}).$$

Remark. In fact, we can naturally identify Gr_G as the étale sheafification of the functor that associates to each perfectoid affinoid algebra (R, R^{+}) over C the coset $G(B_{\mathrm{dR}}(R))/G(B_{\mathrm{dR}}^{+}(R))$.

Proposition 2.1.6 ([SW20, Corollary 19.3.4])**.** *Given $\mu \in X_*(T)^{+}$, there exists a locally spatial diamond $\mathrm{Gr}_{G,\mu}$ with*

$$\mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^{+})\mu(t)^{-1}G(B_{\mathrm{dR}}^{+})/G(B_{\mathrm{dR}}^{+}).$$

Remark. In this paper, we won't use the language of diamonds in an essential way because we are only interested in the C-valued points of Gr_G and $\mathrm{Gr}_{G,\mu}$.

Definition 2.1.7. Let μ be a dominant cocharacter of G.

(1) We refer to the locally spatial diamond $\mathrm{Gr}_{G,\mu}$ in Proposition 2.1.6 as the *Schubert cell* of Gr_G associated to μ.

(2) We define the *parabolic subgroup of G associated to μ* by

$$P_\mu := \{g \in G : \lim_{t\to 0} \mu(t)g\mu(t)^{-1} \text{ exists}\}.$$

(3) We define the *flag variety* associated to the pair (G, μ) by

$$\mathscr{F}\ell(G, \mu) := G/P_\mu.$$

(4) We define the *Bialynicki-Birula map* associated to μ as the map

$$\mathrm{BB}_\mu : \mathrm{Gr}_{G,\mu}(C) \longrightarrow \mathscr{F}\ell(G, \mu)(C)$$

which associates to $g\mu(t)^{-1}G(B_{\mathrm{dR}}^{+}) \in \mathrm{Gr}_{G,\mu}(C)$ the parabolic subgroup $\overline{g}P_\mu\overline{g}^{-1}$, where $\overline{g}$ denotes the image of g under the natural map $G(B_{\mathrm{dR}}^{+}) \to G(C)$.

Proposition 2.1.8 ([CS17, Theorem 3.4.5])**.** *If μ is a minuscule cocharacter of G, the Bialynicki-Birula map BB_μ is bijective.*

2.2. G-bundles on the Fargues-Fontaine curve.

Definition 2.2.1. Fix a uniformizer π of E and a pseudouniformizer ϖ of $C^\flat$. Let q be the number of elements in the residue field of E.

(1) We set
$$\mathcal{Y} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \setminus \{|\pi[\varpi]| = 0\},$$
where we write $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) := W(\mathcal{O}_{C^\flat}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$ for the ring of ramified Witt vectors over $\mathcal{O}_{C^\flat}$ with coefficients in $\mathcal{O}_E$ and the Teichmuller lift $[\varpi]$ of ϖ, and define the *adic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}$$
where ϕ denotes the automorphism of $\mathcal{Y}$ induced by the q-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$.

(2) We define the *schematic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by
$$X := \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})^{\phi = \pi^n}\right).$$

Remark. The definition of the adic Fargues-Fontaine curve relies on the fact that the action of ϕ on $\mathcal{Y}$ is properly discontinuous.

Theorem 2.2.2 ([Ked16, Theorem 4.10], [FF18, Théorème 6.5.2], [KL15, Theorem 8.7.7])**.** *We have the following statements:*

(1) $\mathcal{X}$ is a Noetherian adic space over E.
(2) X is a Dedekind scheme over E.
(3) There exists an equivalence of the categories of vector bundles on $\mathcal{X}$ and X, induced by pullback along a natural map of locally ringed spaces $\mathcal{X} \longrightarrow X$.

Remark. The scheme X is not a curve in the usual sense as it is not of finite type over E.

In light of the statement (3) in Theorem 2.2.2, we will henceforth identify G-bundles on $\mathcal{X}$ with G-bundles on X.

Definition 2.2.3. Given an element $b \in B(G)$, we define the associated G-bundle $\mathcal{E}_b$ on $\mathcal{X}$ (or on X) by descending along the map $\mathcal{Y} \longrightarrow \mathcal{Y}/\phi^{\mathbb{Z}} = \mathcal{X}$ the trivial G-bundle on $\mathcal{Y}$ equipped with the ϕ-linear automorphism given by b.

Theorem 2.2.4 ([Far20, Théorème 5.1])**.** *The map $B(G) \longrightarrow H^1_{\acute{e}t}(X, G)$ sending b to the isomorphism class of $\mathcal{E}_b$ is a bijection.*

Proposition 2.2.5. *The set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X admit a natural bijection which is compatible with direct sums, duals, and ranks.*

Proof. Consider an arbitrary integer $n > 0$. Given $b \in B(\mathrm{GL}_n)$, we write N_b for the isocrystal over $\breve{E}$ with underlying vector space $\breve{E}^{\oplus n}$ and the Frobenius-semilinear automorphism given by b. As observed by Kottwitz [Kot85], there exists a natural bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of isocrystals over $\breve{E}$ of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of N_b. Moreover, Theorem 2.2.4 yields a bijection between $B(\mathrm{GL}_n)$ and the set of isomorphism classes of vector bundles over X of rank n where $b \in B(\mathrm{GL}_n)$ maps to the isomorphism class of $\mathcal{E}_b$. We thus obtain a bijection between the set of isomorphism classes of isocrystals over $\breve{E}$ and the set of isomorphism classes of vector bundles on X. It is straight forward to check that this bijection is compatible with diret sums, duals, and ranks. $\square$

Definition 2.2.6. Let $\mathcal{E}$ be a vector bundle on X. We denote by $N(\mathcal{E})$ the isomorphism class of isocrystals over $\breve{E}$ that corresponds to $\mathcal{E}$ under the bijection in Proposition 2.2.5.

(1) We write $\mathrm{rk}(\mathcal{E})$ for the rank of $\mathcal{E}$, and define the *degree* of $\mathcal{E}$, denoted by $\deg(\mathcal{E})$, to be the degree of $N(\mathcal{E})$.
(2) We define the *Harder-Narasimhan (HN) polygon* of $\mathcal{E}$ by $\mathrm{HN}(\mathcal{E}) := -\mathrm{Newt}(N(\mathcal{E})^\vee)$, where $\mathrm{Newt}(N(\mathcal{E})^\vee)$ refers to the Newton polygon of the dual of $N(\mathcal{E})$.
(3) We say that $\mathcal{E}$ is *semistable* of slope λ if $\mathrm{HN}(\mathcal{E})$ is a line segment of slope λ.

Remark. The definition of $\mathrm{HN}(\mathcal{E})$ is in line with the convention that Newton polygons are convex while Harder-Narasimhan polygons are concave. It is also worthwhile to mention that the correct (or usual) definition of semistability should be given in terms of the Harder-Narasimhan formalism for vector bundles on X; in fact, the equivalence of our definition and the correct definition is due to a highly nontrivial result of Fargues-Fontaine [FF18].

Proposition 2.2.7. *Let $\mathcal{E}$ be a vector bundle on X.*

(1) $\mathcal{E}$ admits a direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ where the $\mathcal{E}_i$'s are semistable vector bundles on X of distinct slopes.
(2) If the $\mathcal{E}_i$'s are arranged in order of descending slope, $\mathrm{HN}(\mathcal{E})$ *is given by the concatenation of the polygons* $\mathrm{HN}(\mathcal{E}_i)$.

Proof. The assertion is evident by Proposition 2.2.5 and the semisimplicity of isocrystals. □

Remark. The statement (2) implies that the direct summands $\mathcal{E}_i$ are uniquely determined up to permutations.

Definition 2.2.8. Let $\mathcal{E}$ be a vector bundle on X. We refer to the direct sum decomposition $\mathcal{E} \simeq \oplus \mathcal{E}_i$ in Proposition 2.2.7 as the *Harder-Narasimhan (HN) decomposition* of $\mathcal{E}$.

2.3. The Newton stratification of Schubert cells and flag varieties.

For the rest of this paper, we fix a closed point ∞ on X given by the following proposition:

Proposition 2.3.1 ([FF18, Théorèmes 6.5.2 and 7.3.3], [CT22, Remark 1.7])**.** *There exists a closed point ∞ on X with the following properties:*

(i) $X - \infty$ is the spectrum of a principal domain $B_e \subseteq B_{\mathrm{dR}}$.
(ii) The completed local ring at ∞ is canonically isomorphic to B_{dR}^+.

Remark. A closed point on X corresponds to a characteristic 0 untilt of $C^\flat$ (i.e., a perfectoid field K with an isomorphism $K^\flat \simeq C^\flat$) up to ϕ-equivalences. We may take ∞ to be the closed point on X corresponding to C with the identity map on $C^\flat$. The field C alone does not determine ∞ as $C^\flat$ has automorphisms which are not ϕ-equivalent to the identity map.

Proposition 2.3.2. *The set $H^1_{\acute{e}t}(X, G)$ is naturally in bijection with the set of isomorphism classes of triples $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, \beta)$ where*

- *$\mathcal{E}^\circ$ is a G-bundle on $X - \infty$,*
- *$\widehat{\mathcal{E}}$ is a trivial G-bundle on* $\mathrm{Spec}\,(B_{\mathrm{dR}}^+)$*, and*
- *β is a gluing map of $\mathcal{E}^\circ$ and $\widehat{\mathcal{E}}$ over* $\mathrm{Spec}\,(B_{\mathrm{dR}})$.

Proof. Every G-bundle on X becomes trivial after the pullback via the map $\mathrm{Spec}\,(B_{\mathrm{dR}}^+) \to X$ induced by ∞, as noted by Nguyen-Viehmann [NV23, §2.1] and Chen-Tong [CT22, Remark 1.7]. Hence the desired assertion follows from Proposition 2.3.1 and the theorem of Beauville-Laszlo [BL95]. □

Definition 2.3.3. Let $\mathcal{E}$ be a G-bundle on X. A *modification* of $\mathcal{E}$ at ∞ is a G-bundle $\mathcal{E}'$ on X together with an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$ on $X - \infty$.

Example 2.3.4. Consider an element $b \in B(G)$ and a point $x \in \mathrm{Gr}_G(C)$. We may write $x = gG(B_{\mathrm{dR}}^+)$ for some $g \in G(B_{\mathrm{dR}})$ under the identification $\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$ noted in Proposition 2.1.5. Now, in light of Proposition 2.3.2 we take a triple $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, \beta)$ corresponding to $\mathcal{E}_b$ and a G-bundle $\mathcal{E}_{b,x}$ on X corresponding to $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, g\beta)$. By construction, $\mathcal{E}_{b,x}$ is naturally a modification of $\mathcal{E}_b$ at ∞.

Definition 2.3.5. Consider an element $b \in B(G)$ and a dominant cocharacter μ of G.

(1) For each $x \in \mathrm{Gr}_G(C)$, we refer to the G-bundle $\mathcal{E}_{b,x}$ constructed in Example 2.3.4 as the *modification of $\mathcal{E}_b$ at ∞ induced by x.*
(2) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to* b in $\mathrm{Gr}_{G,\mu}$ as the subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ with
$$\mathrm{Gr}_{G,\mu,b}^{b'}(C) = \{x \in \mathrm{Gr}_{G,\mu}(C) : \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$
(3) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to* b in $\mathscr{F}\ell(G,\mu)$ as the subvariety $\mathscr{F}\ell(G,\mu,b)^{b'}$ of $\mathscr{F}\ell(G,\mu)$ such that $\mathscr{F}\ell(G,\mu,b)^{b'}(C)$ is the image of $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ under the map BB_μ.

Remark. The subdiamond $\mathrm{Gr}_{G,\mu,b}^{b'}$ of $\mathrm{Gr}_{G,\mu}$ is uniquely determined by its set of C-points since $\mathrm{Gr}_{G,\mu}$ is a locally spatial diamond.

2.4. **Subsheaves and extensions of vector bundles on the Fargues-Fontaine curve.**

Definition 2.4.1. Given two integers n and d with $n > 0$, a *rationally tuplar polygon* of rank n and degree d is the graph $\mathscr{P}$ of a continuous function f with the following properties:

(i) f is defined on $[0, n]$ with $f(0) = 0$ and $f(n) = d$.
(ii) f is linear on $[i-1, i]$ for each $i = 1, \cdots, n$ with a rational slope denoted by $\lambda_i(\mathscr{P})$.

Example 2.4.2. We are particularly interested in the following rationally tuplar polygons:

(1) For every vector bundle $\mathcal{E}$ on X of rank n and degree d, its HN polygon $\mathrm{HN}(\mathcal{E})$ is a rationally tuplar polygon of rank n and degree d.
(2) For $G = \mathrm{GL}_n$ with Borel subgroup B of upper triangular matrices and maximal torus T of diagonal matrices, we regard all dominant cocharacters as rationally tuplar polygons of rank n under the natural identification
$$X_*(T)^+ \cong \{(a_i) \in \mathbb{Z}^n : a_1 \geq a_2 \geq \cdots \geq a_n\}.$$
(3) We write $\underline{d/n}^{(n)}$ for the line segment connecting $(0,0)$ and (n,d), which is a rationally tuplar polygon of rank n and degree d.

Definition 2.4.3. Let $\mathbb{P}_n$ denote the set of rationally tuplar polygons of rank n.

(1) We define the *Bruhat order* $\geq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \geq \mathscr{Q}$ if we have
$$\sum_{i=1}^{j} \lambda_i(\mathscr{P}) \geq \sum_{i=1}^{j} \lambda_i(\mathscr{Q}) \qquad \text{for each } j = 1, \cdots, n$$
with equality for $j = n$.
(2) We define the *slopewise dominance order* $\succeq$ on $\mathbb{P}_n$ by writing $\mathscr{P} \succeq \mathscr{Q}$ if we have $\lambda_i(\mathscr{P}) \geq \lambda_i(\mathscr{Q})$ for each $i = 1, \cdots, n$.

Remark. Intuitively, we have $\mathscr{P} \geq \mathscr{Q}$ if and only if $\mathscr{P}$ lies on or above $\mathscr{Q}$ with the same endpoints, as illustrated by Figure 3.

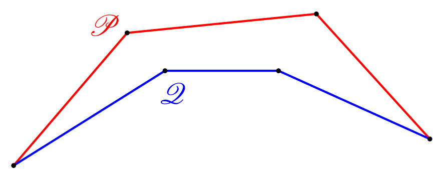

FIGURE 3. Illustration of the Bruhat order

Proposition 2.4.4 ([Hon21, Theorem 1.2.1])**.** *Let $\mathcal{D}$ and $\mathcal{E}$ be vector bundles on X of rank n. Then $\mathcal{D}$ is a subsheaf of $\mathcal{E}$ if and only if we have* $\mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{D})$.

Definition 2.4.5. Given vector bundles $\mathcal{D}, \mathcal{E}$ and $\mathcal{F}$ on X, we define a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$*-permutation* of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$ to be a rationally tuplar polygon $\mathscr{P} \geq \mathrm{HN}(\mathcal{E})$ with the following properties:

(i) The tuple $(\lambda_i(\mathscr{P}))$ is a permutation of the tuple $(\lambda_i(\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})))$.
(ii) For each $i = 1, \cdots, \mathrm{rk}(\mathcal{E})$, we have
  - $\lambda_i(\mathscr{P}) < \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D})$, and
  - $\lambda_i(\mathscr{P}) > \lambda_i(\mathrm{HN}(\mathcal{E}))$ only if $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{F})$.

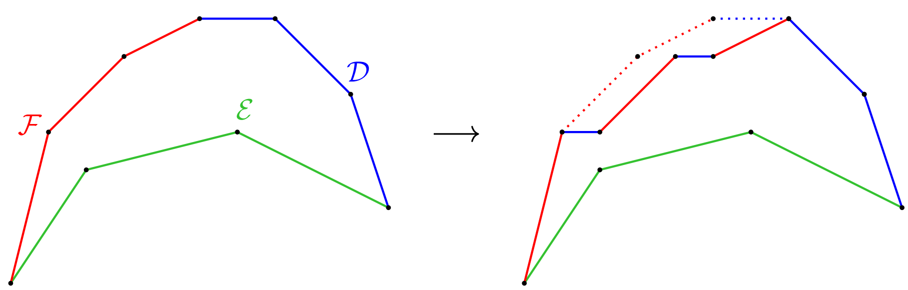

FIGURE 4. Illustration of the conditions in Definition 2.4.5

Proposition 2.4.6 ([FF18, Proposition 5.6.23])**.** *Given vector bundles $\mathcal{D}$ and $\mathcal{F}$ on X such that the minimum slope in* $\mathrm{HN}(\mathcal{D})$ *is greater than or equal to the maximum slope in* $\mathrm{HN}(\mathcal{F})$*, every extension of $\mathcal{F}$ by $\mathcal{D}$ splits.*

Proposition 2.4.7 ([Hon22, Theorem 3.12], [CT22, Proposition 5.3])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X such that there exists a short exact sequence*

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

There exists a $(\mathcal{D}, \mathcal{E}, \mathcal{F})$-permutation of $\mathrm{HN}(\mathcal{D} \oplus \mathcal{F})$.

Proposition 2.4.8 ([Hon22, Theorem 4.4], [CT22, Proposition 5.9])**.** *Let $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on X. We write the HN decomposition of $\mathcal{F}$ as*

$$\mathcal{F} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$

where the $\mathcal{F}_i$'s are arranged in order of descending slope. There exists a short exact sequence

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

if and only if there exists a sequence of vector bundles $\mathcal{D} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}$ on X such that the polygon $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ *has an $(\mathcal{E}_{i-1}, \mathcal{E}_i, \mathcal{F}_i)$-permutation for each $i = 1, \cdots, m$.*

3. Nonempty Newton strata in minuscule Schubert cells for GL_n

In this section, we classify all nonempty Newton strata in an arbitrary minuscule Schubert cell for GL_n by studying modifications of vector bundles on the Fargues-Fontaine curve. We first establish in §3.1 an inductive classification for nonempty Newton strata associated to an arbitrary element of $B(\mathrm{GL}_n)$. We then prove in §3.2 some combinatorial lemmas about rationally tuplar polygons and use them in §3.3 to give an explicit classification of all nonempty Newton strata associated to a large class of element of $B(\mathrm{GL}_n)$. Throughout this section, we take dominant cocharacters of GL_n with respect to the standard Borel subgroup of upper triangular matrices and the standard maximal torus of diagonal matrices.

3.1. An inductive classification of nonempty Newton strata.

Definition 3.1.1. Given a rationally tuplar polygon $\mathscr{P}$ of rank n, we define its *dual* to be the rationally tuplar polygon $\mathscr{P}^*$ with $\lambda_i(\mathscr{P}^*) = -\lambda_{n+1-i}(\mathscr{P})$ for each $i = 1, \cdots, n$.

Example 3.1.2. We illustrate the notion of duality for the polygons in Example 2.4.2.

(1) For a vector bundle $\mathcal{E}$ on X of rank n, we have $\mathrm{HN}(\mathcal{E})^* = \mathrm{HN}(\mathcal{E}^\vee)$ where $\mathcal{E}^\vee$ denotes the dual bundle of $\mathcal{E}$.
(2) For a dominant cocharacter μ of GL_n, the polygon μ^* represents the unique dominant cocharacter in the conjugacy class of μ^{-1}.
(3) For arbitrary integers d and n, we have $\underline{d/n}^{(n)^*} = \underline{-d/n}^{(n)}$.

Proposition 3.1.3 ([CFS21, Proposition 5.2], [Vie24, Corollary 5.4])**.** *Let b and b' be elements of $B(\mathrm{GL}_n)$ such that $\mathcal{E}_b$ is semistable. Given a dominant cocharacter μ of GL_n, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if we have*

$$\nu(b) + \mu^* \geq \nu(b') \tag{3.1}$$

where $\nu(b)$ and $\nu(b')$ respectively denote $\mathrm{HN}(\mathcal{E}_b)$ and $\mathrm{HN}(\mathcal{E}_{b'})$.

Remark. For a reductive group G and a basic element $b \in B(G)$, the results of Chen-Fargues-Shen [CFS21, Proposition 5.2] and Viehmann [Vie24, Corollary 5.4] classify all nonempty newton strata with respect to b in an arbitrary Schubert cell in terms of the Kottwitz map and the Newton map defined by Kottwitz [Kot85]. In our context, their results are translated to Proposition 3.1.3 by the following facts:

(a) An element $b \in B(\mathrm{GL}_n)$ is basic if and only if $\mathcal{E}_b$ is semistable.
(b) The condition involving the Kottwitz map holds for all elements in $B(\mathrm{GL}_n)$.
(c) The condition involving the Newton map is equivalent to the inequality (3.1) as $\nu(b)$ and $\nu(b')$ are identified with the (concave) Newton polygons of b and b'.

Lemma 3.1.4. *Let b be an element of $B(\mathrm{GL}_n)$. For $x = \underline{1}^{(n)}(t)\,\mathrm{GL}_n(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_n,\underline{1}^{(n)}}(C)$, we have $\mathrm{HN}(\mathcal{E}_{b,x}) = \mathrm{HN}(\mathcal{E}_b) - \underline{1}^{(n)}$.*

Proof. Let us write the HN decomposition of $\mathcal{E}_b$ as

$$\mathcal{E}_b \simeq \bigoplus_{i=1}^{m} \mathcal{E}_{b_i} \qquad \text{with } b_i \in B(\mathrm{GL}_{n_i}).$$

For each $i = 1, \cdots, m$, we take $x_i := \underline{1}^{(n_i)}(t)\,\mathrm{GL}_{n_i}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GL}_{n_i},\underline{1}^{(n_i)}}(C)$. Then we have $\mathrm{HN}(\mathcal{E}_{b_i,x_i}) \leq \mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ by Proposition 3.1.3 and thus find $\mathrm{HN}(\mathcal{E}_{b_i,x_i}) = \mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ as $\mathrm{HN}(\mathcal{E}_{b_i}) - \underline{1}^{(n_i)}$ is a line segment. Now the desired assertion follows by the fact that $\mathcal{E}_{b,x}$ is a direct sum of the vector bundles $\mathcal{E}_{b_i,x_i}$. □

Proposition 3.1.5. *Let μ be a dominant cocharacter of GL_n. For elements $b, b' \in B(\mathrm{GL}_n)$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty if and only if it contains a C-point.*

Proof. The assertion is evident by definition. $\square$

Proposition 3.1.6. *Let μ be a dominant cocharacter of GL_n with nonnegative slopes. For two elements $b, b' \in B(\mathrm{GL}_n)$, we have the following equivalent conditions:*

(i) $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ *is nonempty.*

(ii) $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}$ *is nonempty.*

(iii) $\mathrm{Gr}^{\widetilde{b'}}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}$ *is nonempty for $\widetilde{b'} \in B(\mathrm{GL}_n)$ with $\mathrm{HN}(\mathcal{E}_{\widetilde{b'}}) = \mathrm{HN}(\mathcal{E}_{b'}) - \underline{1}^{(n)}$.*

Proof. For $x = g\mu(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, we take $x^* := g^{-1}\mu^*(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu^*}(C)$ and find $\mathcal{E}_{b',x^*} \simeq \mathcal{E}_b$, thereby deducing that x^* lies in $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}(C)$. Similarly, every point in $\mathrm{Gr}^{b}_{\mathrm{GL}_n,\mu^*,b'}(C)$ gives rise to a point in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$. Hence by Proposition 3.1.5 we establish the equivalence of the conditions (i) and (ii).

Now it remains to verify the equivalence of the conditions (i) and (iii). For every $x = g\mu(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, we take $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ and find $\mathcal{E}_{b,\widetilde{x}} \simeq \mathcal{E}_{\widetilde{b'}}$ by Lemma 3.1.4, thereby deducing that $\widetilde{x}$ lies in $\mathrm{Gr}^{\widetilde{b'}}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)},b}(C)$. Conversely, for every $\widetilde{x} := g\mu(t)\underline{1}^{(n)}(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_n,\mu+\underline{1}^{(n)}}(C)$ we take $x := g\mu(t)G(B^+_{\mathrm{dR}}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$ and find $\mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}$ by Lemma 3.1.4, thereby deducing that x lies in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$. Hence we complete the proof by Proposition 3.1.5. $\square$

Remark. In light of Proposition 3.1.6, for our desired classification it suffices to consider minuscule cocharacters with slopes 0 and 1.

Definition 3.1.7. Let $\mathcal{E}$ be a vector bundle on X of rank n.

(1) Given a dominant cocharacter μ of GL_n, we define an *effective modification* of $\mathcal{E}$ at ∞ of *type* μ to be an injective $\mathcal{O}_X$-module map $\mathcal{E}' \hookrightarrow \mathcal{E}$ whose cokernel is the skyscraper sheaf at ∞ with value $\displaystyle\bigoplus_{i=1}^{n} B^+_{\mathrm{dR}}/t^{\lambda_i(\mu)}B^+_{\mathrm{dR}}$.

(2) We say that an effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ is *minuscule of degree d* if its type is minuscule of degree d with slopes 0 and 1.

Proposition 3.1.8. *Take a dominant cocharacter μ of GL_n and two elements $b, b' \in B(\mathrm{GL}_n)$.*

(1) If μ has nonnegative slopes, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ.

(2) If μ is minuscule with slopes 0 and 1, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exists a minuscule effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞.

Proof. As the second statement is a special case of the first statement, it suffices to prove the first statement. If $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is not empty, Proposition 3.1.5 yields a point $x \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$, which gives rise to an effective modification $\mathcal{E}_{b,x} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Let us now assume for the converse that there exists an effective modification $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ at ∞ of type μ. Take triples $(\mathcal{E}^\circ_b, \widehat{\mathcal{E}_b}, \beta_b)$ and $(\mathcal{E}^\circ_{b'}, \widehat{\mathcal{E}_{b'}}, \beta_{b'})$ which respectively correspond to $\mathcal{E}_b$ and $\mathcal{E}_{b'}$ under the bijection in Proposition 2.3.2. We may set $\mathcal{E}^\circ_b = \mathcal{E}^\circ_{b'}$ since the map $\mathcal{E}_{b'} \hookrightarrow \mathcal{E}_b$ is an isomorphism on $X - \infty$. Then we conjugate β_b by a suitable element in $\mathrm{GL}_n(B^+_{\mathrm{dR}})$ to write $\beta_{b'} = g\mu(t)\beta_b$ for some $g \in \mathrm{GL}_n(B^+_{\mathrm{dR}})$, and in turn find $g\mu(t)\,\mathrm{GL}_n(B^+_{\mathrm{dR}}) \in \mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}(C)$ to complete the proof. $\square$

Proposition 3.1.9. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n. Take a direct sum decomposition*

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.2}$$

such that $\mathrm{HN}(\mathcal{D})$ *coincides with the line segment of maximal slope in* $\mathrm{HN}(\mathcal{E})$. *There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence*

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Proof. The assertion is essentially a result of Chen-Tong [CT22, Proposition 4.6]. Our main observation is that, while the result in loc. cit. for $G = \mathrm{GL}_n$ only concerns the case where $\mathcal{E}$ is semistable, its proof remains valid without the semistability assumption on $\mathcal{E}$. For convenience of the readers, we explain how the result in loc. cit. is translated to the desired assertion.

Let us take $b, b' \in B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$ and $\mathcal{E}' \simeq \mathcal{E}_{b'}$. We write r for the rank of $\mathcal{D}$ and P for the standard parabolic subgroup of GL_n with Levi subgroup

$$M := \mathrm{GL}_r \times \mathrm{GL}_{n-r} \subseteq \mathrm{GL}_n\,.$$

The direct sum decomposition (3.2) corresponds to an element $b_M \in B(M)$ which maps to b under the natural map $B(M) \longrightarrow B(G)$. Let $E(M, b')$ denote the set of all elements $b'_M \in B(M)$ which correspond to a direct sum $\mathcal{D}' \oplus \mathcal{F}'$ for some vector bundles $\mathcal{D}'$ of rank r and $\mathcal{F}'$ of rank $n-r$ such that $\mathcal{E}' \simeq \mathcal{E}_{b'}$ arises as an extension of $\mathcal{F}'$ by $\mathcal{D}'$.

We take μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. In addition, we choose an arbitrary element w in the Weyl group of GL_n and denote by μ^w the dominant cocharacter of M whose M-conjugacy class contains the w-conjugate of μ. We have $\mu^w = (\mu_1, \mu_2)$ for some minuscule dominant cocharacters μ_1 of GL_r and μ_2 of GL_{n-r}. We denote the degrees of μ_1 and μ_2 respectively by d_1 and d_2.

Let $\mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w$ be the subscheme of $\mathscr{F}\ell(\mathrm{GL}_n, \mu)$ given by the P-orbit of P_{μ^w}. The projection to M induces a map

$$\mathrm{pr}_{P,w} : \mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w \longrightarrow \mathscr{F}\ell(M, \mu^w).$$

The aforementioned result of Chen-Tong [CT22, Proposition 4.6] yields an identity

$$\mathrm{pr}_{P,w}\left(\mathscr{F}\ell(\mathrm{GL}_n, \mu)_P^w \cap \mathscr{F}\ell(\mathrm{GL}_n, \mu, b)^{b'}\right) = \bigsqcup_{b'_M \in E(M,b')} \mathscr{F}\ell(M, \mu^w, b_M)^{b'_M}. \tag{3.3}$$

As both μ and μ^w are minuscule, Proposition 2.1.8 implies that the Newton strata on $\mathrm{Gr}_{\mathrm{GL}_n,\mu}$ and Gr_{M,μ^w} are respectively identified with the Newton strata on $\mathscr{F}\ell(\mathrm{GL}_n, \mu)$ and $\mathscr{F}\ell(M, \mu^w)$. Hence the identity (3.3) shows that for minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ of degrees d_1 and d_2 we have the following equivalent conditions:

(i) There exists a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{F} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{D}' \arrow[r] \arrow[u, hook, "\alpha"] & \mathcal{E}' \arrow[r] \arrow[u, hook] & \mathcal{F}' \arrow[r] \arrow[u, hook, "\beta"] & 0 \end{tikzcd} \tag{3.4}$$

with the top row given by the direct sum decomposition (3.2) and the middle vertical arrow being a minuscule effective modification at ∞ (of degree d).

(ii) There exists a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Since w is arbitrary, we deduce the desired assertion. $\square$

Remark. The necessity part of Proposition 3.1.9 is evident as every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a commutative diagram (3.4). The main point of Proposition 3.1.9 is the sufficiency part, which is essentially equivalent to the identity (3.3) by Chen-Tong [CT22].

Proposition 3.1.10 ([FF18, §5.5.2.1])**.** *Let $\mathcal{E}$ be a vector bundle on X. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, its degree is equal to $\deg(\mathcal{E}) - \deg(\mathcal{E}')$.*

Lemma 3.1.11. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n such that $\mathcal{E}$ is semistable. Take μ to be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following equivalent inequalities:*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}'). \tag{3.5}$$

Proof. By Proposition 3.1.3, Proposition 3.1.8 and Proposition 3.1.10, there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the first inequality in (3.5). If we write λ for the slope of the line segment $\mathrm{HN}(\mathcal{E})$, the polygon $\mathrm{HN}(\mathcal{E}) + \mu^*$ has two distinct slopes λ and $\lambda - 1$. Hence it is not hard to verify the equivalence of the two inequalities in (3.5) by the concavity of HN polygons, thereby deducing the desired assertion. □

Theorem 3.1.12. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Consider two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$. Take a direct sum decomposition*

$$\mathcal{E}_b \simeq \mathcal{E}_a \oplus \mathcal{E}_c \quad \text{with} \quad a \in B(\mathrm{GL}_r) \text{ and } c \in B(\mathrm{GL}_{n-r})$$

such that $\mathrm{HN}(\mathcal{E}_a)$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E}_b)$.

(1) If the degree of μ is not equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$, the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is empty.

(2) If the degree of μ is equal to $\deg(\mathcal{E}_b) - \deg(\mathcal{E}_{b'})$ the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ is nonempty if and only if there exist $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$ with the following properties:

(i) We have $\mathrm{HN}(\mathcal{E}_{a'}) + \underline{1}^{(r)} \succeq \mathrm{HN}(\mathcal{E}_a) \succeq \mathrm{HN}(\mathcal{E}_{a'})$.

(ii) If we write the HN decomposition of $\mathcal{E}_{c'}$ as

$$\mathcal{E}_{c'} \simeq \bigoplus_{i=1}^{m} \mathcal{F}_i$$

where $\mathcal{F}_i$ are arranged in order of descending slope, there exists a sequence of vector bundles $\mathcal{E}_{a'} = \mathcal{E}_0, \mathcal{E}_1, \cdots, \mathcal{E}_m = \mathcal{E}_b$ on X such that $\mathrm{HN}(\mathcal{E}_{i-1} \oplus \mathcal{F}_i)$ has an $(\mathcal{E}_{i-1}, \mathcal{E}_i, \mathcal{F}_i)$-permutation for each $i = 1, \cdots, r$.

(iii) The Newton stratum $\mathrm{Gr}^{c'}_{\mathrm{GL}_n,\overline{\mu},c}$ is nonempty where $\overline{\mu}$ is a minuscule dominant cocharacter of GL_{n-r} of degree $\overline{d} := \deg(\mathcal{E}_c) - \deg(\mathcal{E}_{c'})$ with slopes 0 and 1.

Proof. The assertion is straightforward to verify by Proposition 2.4.8, Proposition 3.1.8, Proposition 3.1.9, Proposition 3.1.10, and Lemma 3.1.11. □

Remark. The elements $a \in B(\mathrm{GL}_r)$ and $c \in B(\mathrm{GL}_{n-r})$ are uniquely determined by the HN decomposition of $\mathcal{E}_b$. In addition, the Schubert cell $\mathrm{Gr}_{\mathrm{GL}_n,\overline{\mu}}$ contains finitely many nonemtpy Newton strata, as easily seen by Proposition 2.4.4 and Proposition 3.1.8. Hence the conditions (i) and (iii) together yield finitely many candidates for $a' \in B(\mathrm{GL}_r)$ and $c' \in B(\mathrm{GL}_{n-r})$. We can thus use Theorem 3.1.12 to inductively classify all nonempty Newton strata in an arbitrary minuscule Schubert cell of $\mathrm{Gr}_{\mathrm{GL}_n}$.

3.2. Concave rationally tuplar polygons.

Definition 3.2.1. Given a rationally tuplar polygon $\mathscr{P}$, we define its *concave rearrangement* to be the rationally tuplar polygon $\widehat{\mathscr{P}}$ such that the tuple $(\lambda_i(\widehat{\mathscr{P}}))$ is the rearrangement of $(\lambda_i(\mathscr{P}))$ in descending order.

Lemma 3.2.2. *For every rationally tuplar polygon $\mathscr{P}$, we have $\widehat{\mathscr{P}} \geq \mathscr{P}$.*

Proof. The assertion is evident by definition. □

Remark. In fact, $\widehat{\mathscr{P}}$ is the maximal rearrangement of $\mathscr{P}$ with respect to the Bruhat order.

Definition 3.2.3. Given two rationally tuplar polygon $\mathscr{P}$ and $\mathscr{Q}$, we define their direct sum $\mathscr{P} \oplus \mathscr{Q}$ to be the concave rearrangement of the concatenation of $\mathscr{P}$ and $\mathscr{Q}$.

Example 3.2.4. Let us record some important examples of direct sums for our purpose.

(1) For two vector bundles $\mathcal{E}$ and $\mathcal{F}$ on X, we have $\mathrm{HN}(\mathcal{E} \oplus \mathcal{F}) = \mathrm{HN}(\mathcal{E}) \oplus \mathrm{HN}(\mathcal{F})$.

(2) For two minuscule dominant cocharacters μ_1 of GL_{n_1} and μ_2 of GL_{n_2} with slopes 0 and 1, their direct sum (as a rationally tuplar polygon) is a minuscule dominant cocharacter of $\mathrm{GL}_{n_1+n_2}$ with slopes 0 and 1.

Lemma 3.2.5. *Given concave rationally tuplar polygons $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ with $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, we have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$.*

Proof. Let m and n respectively denote the ranks of $\mathscr{P}$ and $\mathscr{Q}$. Take two sets A and B which form a partition of the set $\{1, \cdots, m+n\}$ with

$$(\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in A} = (\lambda_i(\mathscr{P}')) \quad \text{and} \quad (\lambda_i(\mathscr{P}' \oplus \mathscr{Q}'))_{i \in B} = (\lambda_i(\mathscr{Q}')).$$

Let $\mathscr{R}$ to be the rationally tuplar polygon of rank $m+n$ with

$$(\lambda_i(\mathscr{R}))_{i \in A} = (\lambda_i(\mathscr{P})) \quad \text{and} \quad (\lambda_i(\mathscr{R}))_{i \in B} = (\lambda_i(\mathscr{Q})).$$

Since $\mathscr{P}, \mathscr{P}', \mathscr{Q}$ and $\mathscr{Q}'$ are all concave, the inequalities $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$ together imply $\mathscr{R} \geq \mathscr{P}' \oplus \mathscr{Q}'$. Now we find $\mathscr{P} \oplus \mathscr{Q} = \widehat{\mathscr{R}} \geq \mathscr{R}$ by Lemma 3.2.2 to complete the proof. □

Remark. Lemma 3.2.5 does not hold without the concavity assumption. For example, if we take $\mathscr{P} = \mathscr{Q} = \overline{d/r^{(r)}}$ for some integers r and d with $r > 0$, for arbitrary nonlinear convex polygons $\mathscr{P}'$ and $\mathscr{Q}'$ of rank r and degree d we do not have $\mathscr{P} \oplus \mathscr{Q} \geq \mathscr{P}' \oplus \mathscr{Q}'$ despite having $\mathscr{P} \geq \mathscr{P}'$ and $\mathscr{Q} \geq \mathscr{Q}'$, as illustrated in Figure 5.

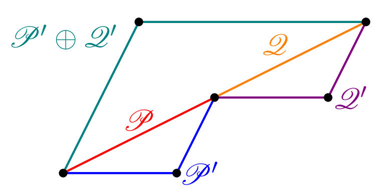

FIGURE 5. A counter example for Lemma 3.2.5 without the concavity assumption

Lemma 3.2.6. *Let $\mathscr{P}$ and $\mathscr{Q}$ be rationally tuplar polygons of rank m and n. For arbitrary rationally tuplar polygons $\mathscr{P}'$ of rank m and $\mathscr{Q}'$ of rank n, we have*

$$(\mathscr{P} \oplus \mathscr{Q}) + (\mathscr{P}' \oplus \mathscr{Q}') \geq (\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}').$$

Proof. We observe that there exist permutations σ and σ' of the set $\{1, \cdots, m+n\}$ with

$$\lambda_i((\mathscr{P} + \mathscr{P}') \oplus (\mathscr{Q} + \mathscr{Q}')) = \lambda_{\sigma(i)}(\mathscr{P} \oplus \mathscr{Q}) + \lambda_{\sigma'(i)}(\mathscr{P}' \oplus \mathscr{Q}') \quad \text{for each } i = 1, \cdots, m+n,$$

and consequently deduce the desired assertion by the concavity of $\mathscr{P} \oplus \mathscr{Q}$ and $\mathscr{P}' \oplus \mathscr{Q}'$. □

3.3. An explicit classification of nonempty Newton strata.

Lemma 3.3.1. *Let $\mathcal{E}$ be a vector bundle on X of rank n. Every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ gives rise to a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ with $\mathrm{HN}(\widetilde{\mathcal{E}}) = \mathrm{HN}(\mathcal{E}) - \underline{1}^{(n)}$.*

Proof. Let μ be the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. Take elements b, b' and $\widetilde{b}$ in $B(\mathrm{GL}_n)$ with $\mathcal{E} \simeq \mathcal{E}_b$, $\mathcal{E}' \simeq \mathcal{E}_{b'}$ and $\widetilde{\mathcal{E}} \simeq \mathcal{E}_{\widetilde{b}}$. The effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ yields a point in $\mathrm{Gr}^{b'}_{\mathrm{GL}_n,\mu,b}$ by Proposition 3.1.8 and Proposition 3.1.10, and in turn yields a point in $\mathrm{Gr}^{\widetilde{b}}_{\mathrm{GL}_n,\mu^*+\underline{1}^{(n)},b'}$ by Proposition 3.1.6. Hence we obtain a minuscule effective modification $\widetilde{\mathcal{E}} \hookrightarrow \mathcal{E}'$ at ∞ by Proposition 3.1.8 as desired. □

Proposition 3.3.2. *Let $\mathcal{E}$ be a vector bundle on X of rank n. For every minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞, we have*

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$$

where μ is the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1.

Proof. The second inequality is an immediate consequence of Proposition 2.4.4 and Lemma 3.3.1. Hence it remains to establish the first inequality. Let us write m for the number of distinct slopes in $\mathrm{HN}(\mathcal{E})$ and proceed by induction on m. If $\mathcal{E}$ is semistable, the assertion is evident by Lemma 3.1.11. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. The numbers of distinct slopes in $\mathrm{HN}(\mathcal{D})$ and $\mathrm{HN}(\mathcal{F})$ are respectively 1 and $m-1$. Now Proposition 3.1.9 yields minuscule effective modifications $\alpha : \mathcal{D}' \hookrightarrow \mathcal{D}$ and $\beta : \mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.6}$$

Let us denote the types of α and β respectively by μ_1 and μ_2. In a concrete form, we have

$$\mu_1 = \underline{1}^{(d_1)} \oplus \underline{0}^{(n_1-d_1)} \quad \text{and} \quad \mu_2 = \underline{1}^{(d_2)} \oplus \underline{0}^{(n_2-d_2)}$$

where we set $n_1 := \mathrm{rk}(\mathcal{D}) = \mathrm{rk}(\mathcal{D}')$, $n_2 := \mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}')$, $d_1 := \deg(\mathcal{D}) - \deg(\mathcal{D}')$ and $d_2 := \deg(\mathcal{F}) - \deg(\mathcal{F}')$. By the induction hypothesis, the minuscule effective modifications α and β at ∞ respectively yield the inequalities

$$\mathrm{HN}(\mathcal{D}) + \mu_1^* \geq \mathrm{HN}(\mathcal{D}') \quad \text{and} \quad \mathrm{HN}(\mathcal{F}) + \mu_2^* \geq \mathrm{HN}(\mathcal{F}').$$

Then by Example 3.2.4, Lemma 3.2.5 and Lemma 3.2.6 we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* = (\mathrm{HN}(\mathcal{D}) \oplus \mathrm{HN}(\mathcal{F})) + (\mu_1^* \oplus \mu_2^*) \geq (\mathrm{HN}(\mathcal{D}) + \mu_1^*) \oplus (\mathrm{HN}(\mathcal{F}) + \mu_2^*) \geq \mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}').$$

In addition, by Proposition 2.4.7 the short exact sequence (3.6) yields the inequality

$$\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}') \geq \mathrm{HN}(\mathcal{E}').$$

We thus obtain the first inequality, thereby completing the proof. □

Remark. The two inequalities in Proposition 3.3.2 are not equivalent in general, although they are equivalent if $\mathcal{E}$ is semistable as shown in Lemma 3.1.11.

Example 3.3.3. Let us present an example showing that the converse of Proposition 3.3.2 does not hold. Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\mathrm{HN}(\mathcal{E}) = \underline{4/3}^{(3)} \oplus \underline{3/4}^{(4)} \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') = \underline{1}^{(2)} \oplus \underline{1/3}^{(3)} \oplus \underline{0}^{(2)}.$$

By construction, we have $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{E}') = 7$, $\deg(\mathcal{E}) = 7$ and $\deg(\mathcal{E}') = 3$. Now for the minuscule dominant cocharacter μ of GL_7 of degree 4 with slopes 0 and 1, we find

$$\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}') \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') + \underline{1}^{(7)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}').$$

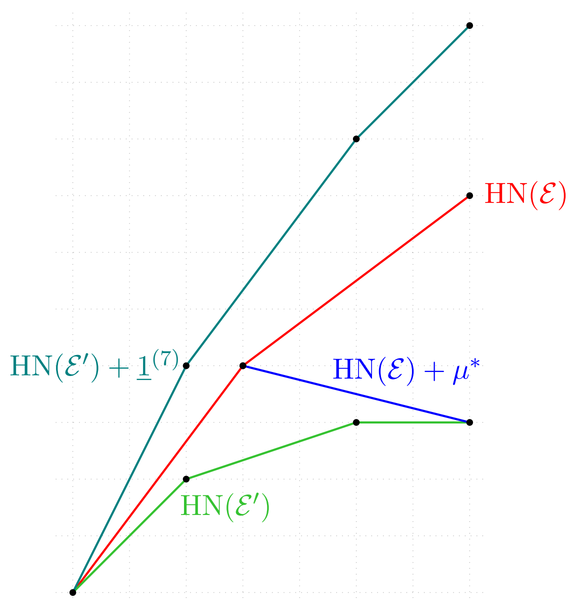

FIGURE 6. A counter example for the converse of Proposition 3.3.2

We wish to show that there does not exist a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Suppose for contradiction that such a modification exists. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}$$

with $\mathrm{HN}(\mathcal{D}) = \underline{4/3}^{(3)}$ and $\mathrm{HN}(\mathcal{F}) = \underline{3/4}^{(4)}$. Proposition 3.1.9 yields minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Then by Proposition 2.4.7 we obtain a $(\mathcal{D}', \mathcal{E}', \mathcal{F}')$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}')$. Since we have $\mathscr{P} \geq \mathrm{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_1(\mathscr{P}) \geq \lambda_1(\mathrm{HN}(\mathcal{E}')) = 1 \quad \text{and} \quad \lambda_2(\mathscr{P}) \geq \lambda_2(\mathrm{HN}(\mathcal{E}')) = 1. \tag{3.7}$$

Moreover, as $\mathcal{F}'$ is a subsheaf of $\mathcal{F}$ by construction, Proposition 2.4.4 implies that all slopes in $\mathrm{HN}(\mathcal{F}')$ are less than or equal to $3/4$. We then deduce by (3.7) that $\lambda_1(\mathscr{P})$ and $\lambda_2(\mathscr{P})$ should occur as a slope of $\mathcal{D}'$, and in turn find that the inequalities in (3.7) are in fact equalities. Therefore $\mathrm{HN}(\mathcal{D}')$ must contain the line segment $\underline{1}^{(2)}$, and consequently is given by $\underline{1}^{(2)} \oplus \underline{d}^{(1)}$ for some integer d. Then we have $d = \lambda_i(\mathscr{P})$ for some $i > 2$ and thus find $d \leq \lambda_i(\mathrm{HN}(\mathcal{E}')) \leq 1/3$. On the other hand, since $\mathcal{D}'$ occurs as a minuscule effective modification of $\mathcal{D}$ at C, Proposition 3.3.2 implies $d \geq 1/3$. Now we have a desired contradiction as d is an integer with $d \leq 1/3$ and $d \geq 1/3$.

Proposition 3.3.4. *Let $\mathcal{E}$ and $\mathcal{E}'$ be vector bundles on X of rank n. Denote by μ the minuscule dominant cocharacter of GL_n of degree $d := \deg(\mathcal{E}) - \deg(\mathcal{E}')$ with slopes 0 and 1. Assume that $\mathcal{E}$ satisfies the following property:*

- *$(*)$ All distinct slopes in $\mathrm{HN}(\mathcal{E})$ differ by more than 1.*

There exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ if and only if $\mathcal{E}$ and $\mathcal{E}'$ satisfy the following conditions:

- *(i) We have $\mathrm{HN}(\mathcal{E}) + \mu^* \geq \mathrm{HN}(\mathcal{E}')$ and $\mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E}) \succeq \mathrm{HN}(\mathcal{E}')$.*
- *(ii) For each breakpoint of $\mathrm{HN}(\mathcal{E})$, there exists a breakpoint of $\mathrm{HN}(\mathcal{E}')$ with the same x-coordinate.*

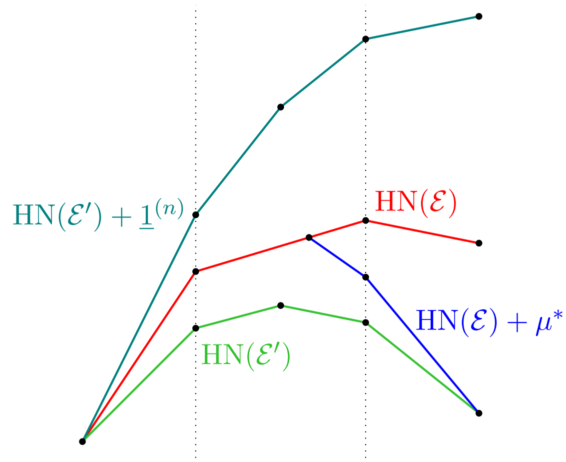

FIGURE 7. Illustration of the conditions in Proposition 3.3.4

Proof. Let us first assume that $\mathcal{E}$ and $\mathcal{E}'$ satisfy the conditions (i) and (ii). We write the HN decomposition of $\mathcal{E}$ as

$$\mathcal{E} \simeq \bigoplus_{i=1}^{m} \mathcal{E}_i \tag{3.8}$$

where the direct summands $\mathcal{E}_i$ are arranged in order of descending slope, and set

$$x_i := \sum_{j=1}^{i} \mathrm{rk}(\mathcal{E}_j) \quad \text{for } i = 0, \cdots, m.$$

By the condition (ii), we get a direct sum decomposition

$$\mathcal{E}' \simeq \bigoplus_{i=1}^{m} \mathcal{E}'_i \tag{3.9}$$

where each $\mathrm{HN}(\mathcal{E}'_i)$ coincides with the restriction of $\mathrm{HN}(\mathcal{E}')$ on the interval $[x_{i-1}, x_i]$. Then by the condition (i) we find

$$\mathrm{HN}(\mathcal{E}'_i) + \underline{1}^{(x_i - x_{i-1})} \succeq \mathrm{HN}(\mathcal{E}_i) \succeq \mathrm{HN}(\mathcal{E}'_i) \quad \text{for } i = 1, \cdots, m.$$

Now for each $i = 1, \cdots, m$, Lemma 3.1.11 yields a minuscule effective modification $\mathcal{E}'_i \hookrightarrow \mathcal{E}_i$ at ∞ as $\mathcal{E}_i$ is semistable. Hence we obtain a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ from the direct sum decompositions (3.8) and (3.9).

For the converse, we now assume that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Since $\mathcal{E}$ and $\mathcal{E}'$ satisfy the condition (i) by Proposition 3.3.2, it remains to establish the condition (ii). We proceed by induction on the number m of distinct slopes in $\mathrm{HN}(\mathcal{E})$. If $\mathcal{E}$ is semistable, the assertion is vacuously true as $\mathrm{HN}(\mathcal{E})$ does not have a breakpoint. We henceforth assume that $\mathcal{E}$ is not semistable, so that we have $m > 1$. Take a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F} \tag{3.10}$$

such that $\mathrm{HN}(\mathcal{D})$ coincides with the line segment of maximal slope in $\mathrm{HN}(\mathcal{E})$. Let us denote the slope of $\mathrm{HN}(\mathcal{D})$ by λ. By construction, $\mathrm{HN}(\mathcal{F})$ has $m-1$ distinct slopes which are all less than $\lambda - 1$ by the property $(*)$. In addition, we have $\mathrm{HN}(\mathcal{E}') + \underline{1}^{(n)} \succeq \mathrm{HN}(\mathcal{E})$ by Proposition 3.3.2 and thus find

$$\lambda_i(\mathrm{HN}(\mathcal{E}')) \geq \lambda - 1 \quad \text{ for } i = 1, \cdots, \mathrm{rk}(\mathcal{D}). \tag{3.11}$$

Now we note by Proposition 3.1.9 that there exist minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ with a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0. \tag{3.12}$$

Then we find

$$\lambda_i(\mathrm{HN}(\mathcal{F}')) \leq \lambda_i(\mathrm{HN}(\mathcal{F})) < \lambda - 1 \quad \text{ for } i = 1, \cdots, \mathrm{rk}(\mathcal{F}') \tag{3.13}$$

by Proposition 2.4.4, and also obtain a $(\mathcal{D}', \mathcal{E}', \mathcal{F}')$-permutation $\mathscr{P}$ of $\mathrm{HN}(\mathcal{D}' \oplus \mathcal{F}')$ by Proposition 2.4.7. For each $i = 1, \cdots, \mathrm{rk}(\mathcal{D})$, the inequalities (3.11) and (3.13) together imply that $\lambda_i(\mathscr{P})$ occurs as a slope in $\mathrm{HN}(\mathcal{D}')$. Since we have $\mathscr{P} \geq \mathrm{HN}(\mathcal{E}')$ by construction, we find

$$\lambda_i(\mathscr{P}) = \lambda_i(\mathrm{HN}(\mathcal{D}')) = \lambda_i(\mathrm{HN}(\mathcal{E}')) \quad \text{ for } i = 1, \cdots, \mathrm{rk}(\mathcal{D})$$

and consequently deduce from the inequalities (3.11) and (3.13) that all slopes in $\mathrm{HN}(\mathcal{D}')$ are greater than all slopes in $\mathrm{HN}(\mathcal{F}')$. Hence the short exact sequence (3.12) induces a direct sum

$$\mathcal{E}' \simeq \mathcal{D}' \oplus \mathcal{F}' \tag{3.14}$$

by Proposition 2.4.6, and consequently yields a breakpoint of $\mathrm{HN}(\mathcal{E}')$ with x-coordinate $\mathrm{rk}(\mathcal{D}') = \mathrm{rk}(\mathcal{D})$. In addition, since we have a minuscule effective modification $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞, we find by the induction hypothesis that for every breakpoint of $\mathrm{HN}(\mathcal{F})$ there exists a breakpoint of $\mathrm{HN}(\mathcal{F}')$ with the same x-coordinate. We thus establish the condition (ii) by the direct sum decompositions (3.10) and (3.14), thereby completing the proof. □

Theorem 3.3.5. *Let μ be a minuscule dominant cocharacter of GL_n with slopes 0 and 1. Take two arbitrary elements $b, b' \in B(\mathrm{GL}_n)$ and write $\nu(b) := \mathrm{HN}(\mathcal{E}_b)$ and $\nu(b') := \mathrm{HN}(\mathcal{E}_{b'})$. Assume that b satisfies the following property:*

$(*)$ All distinct slopes in $\nu(b)$ differ by more than 1.

The Newton stratum $\mathrm{Gr}_{\mathrm{GL}_n,\mu,b}^{b'}$ is nonempty if and only if $\nu(b)$ and $\nu(b')$ satisfy the following conditions:

(i) We have $\nu(b) + \mu^* \geq \nu(b')$ and $\nu(b') + \underline{1}^{(n)} \succeq \nu(b) \succeq \nu(b')$.

(ii) For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.

Proof. The assertion is an immediate consequence of Proposition 3.1.8, Proposition 3.1.10 and Proposition 3.3.4. □

Remark. Theorem 3.3.5 is identical to Theorem 1.1.1. For a non-minuscule cocharacter μ of GL_n with slopes in $[0, d]$, we should be able to get a similar classification theorem with d in place of 1 using the Demazure resolution.

Example 3.3.6. Let us provide an example to show that Proposition 3.3.4 and Theorem 3.3.5 do not hold without assuming $(*)$. Take $\mathcal{E}$ and $\mathcal{E}'$ to be vector bundles on X with

$$\mathrm{HN}(\mathcal{E}) = \underline{5/4}^{(4)} \oplus \underline{3/4}^{(4)} \quad \text{and} \quad \mathrm{HN}(\mathcal{E}') = \underline{3/5}^{(5)} \oplus \underline{1/3}^{(3)}.$$

Then $\mathrm{HN}(\mathcal{E})$ and $\mathrm{HN}(\mathcal{E}')$ do not have breakpoints with the same x-coordinates. We wish to show that there exists a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞. Take vector bundles $\mathcal{D}$, $\mathcal{D}'$, $\mathcal{F}$ and $\mathcal{F}'$ on X with

$$\mathrm{HN}(\mathcal{D}) = \underline{5/4}^{(4)}, \quad \mathrm{HN}(\mathcal{D}') = \underline{1/4}^{(4)}, \quad \mathrm{HN}(\mathcal{F}) = \mathrm{HN}(\mathcal{F}') = \underline{3/4}^{(4)}.$$

By construction, we have a direct sum decomposition

$$\mathcal{E} \simeq \mathcal{D} \oplus \mathcal{F}.$$

In addition, we obtain minuscule effective modifications $\mathcal{D}' \hookrightarrow \mathcal{D}$ and $\mathcal{F}' \hookrightarrow \mathcal{F}$ at ∞ by Lemma 3.1.11, and find a short exact sequence

$$0 \longrightarrow \mathcal{D}' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{F}' \longrightarrow 0$$

by Proposition 2.4.8. Therefore Proposition 3.1.9 yields a minuscule effective modification $\mathcal{E}' \hookrightarrow \mathcal{E}$ at ∞ as desired.

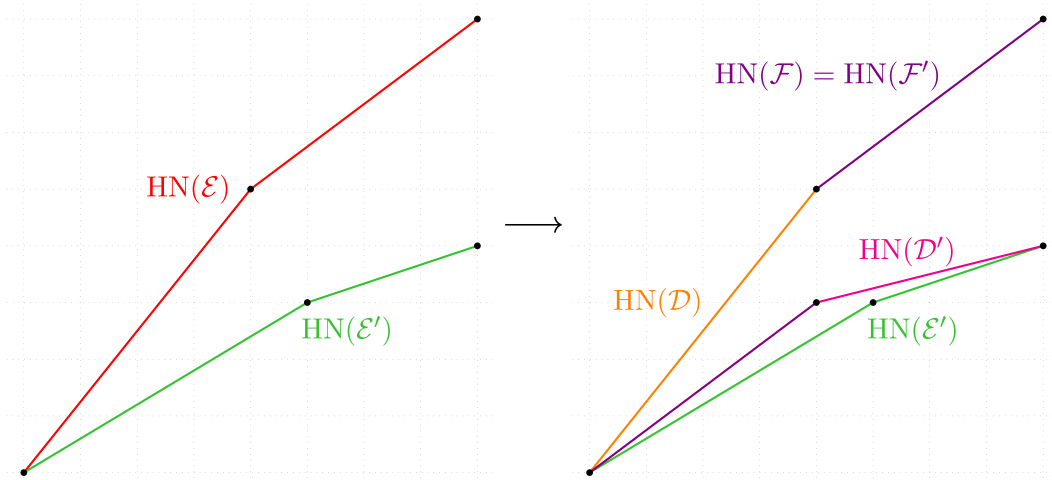

FIGURE 8. Illustration of Example 3.3.6

REFERENCES

[BL95] Arnaud Beauville and Yves Laszlo, *Un lemme de descente*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), no. 1, 335–340.

[CFS21] Miaofen Chen, Laurent Fargues, and Xu Shen, *On the structure of some p-adic period domains*, Camb. J. Math. **9** (2021), no. 1, 213–267.

[Che20] Miaofen Chen, *Fargues-Rapoport conjecture for p-adic period domains in the non-basic case*, J. Eur. Math. Soc., to appear.

[CS17] Ana Caraiani and Peter Scholze, *On the generic part of the cohomology of compact unitary Shimura varieties*, Annals of Math. **186** (2017), no. 3, 649–766.

[CT22] Miaofen Chen and Jilong Tong, *Weakly admissible locus and Newton stratification in p-adic Hodge theory*, Amer. J. Math., to appear.

[Far20] Laurent Fargues, *G-torseurs en théorie de Hodge p-adique*, Comp. Math. **156** (2020), no. 10, 2076–2110.

[FF18] Laurent Fargues and Jean-Marc Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, Astérisque **406** (2018).

[Fon82] Jean-Marc Fontaine, *Sur certains types de representations p-adiques du groupe de Galois d'un corps local; Construction d'un anneau de Barsotti-Tate*, Annals of Mathematics **115** (1982), no. 3, 529–577.

[FS21] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459.

[Hon21] Serin Hong, *Classification of subbundles on the Fargues-Fontaine curve*, Algebra & Number Theory **15** (2021), no. 5, 1127–1156.

[Hon22] ———, *Extensions of vector bundles on the Fargues-Fontaine curve II*, J. Algebraic Geom., to appear.

[Ked16] Kiran S. Kedlaya, *Noetherian properties of Fargues-Fontaine curves*, Int. Math. Res. Not. IMRN (2016), no. 8, 2544–2567.

[KL15] Kiran S. Kedlaya and Ruochuan Liu, *Relative p-adic Hodge theory: Foundations*, Astérisque **371** (2015).

[Kot85] Robert E. Kottwitz, *Isocrystals with additional structure*, Comp. Math. **56** (1985), 201–220.

[NV23] Kieu Hieu Nguyen and Eva Viehmann, *A Harder-Narasimhan stratification of the B_{dR}^{+}-Grassmannian*, Comp. Math. **159** (2023), no. 4, 711–745.

[Rap18] Michael Rapoport, *Accessible and weakly accessible period domains*, Annales de l'ENS **51** (2018), 856–863, appendix to *On the p-adic cohomology of the Lubin-Tate tower* by P. Scholze.

[She23] Xu Shen, *Harder-Narasimhan strata and p-adic period domains*, Trans. Am. Math. Soc **376** (2023), no. 5, 3191–3263.

[SW20] Peter Scholze and Jared Weinstein, *Berkeley lectures on p-adic geometry*, Annals of Math. Studies, vol. 207, 2020.

[Vie24] Eva Viehmann, *On Newton strata in the B_{dR}^{+}-Grassmannian*, Duke Math. J. **173** (2024), no. 1, 177–225.

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